**53. Multiple Choice** If $f(x) = \cos x$, then the Mean Value Theorem guarantees that somewhere between 0 and $\pi/3$, $f'(x) =$

(A) $-\frac{3}{2\pi}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $-1$ (D) 0 (E) $\frac{1}{2}$

**54. Multiple Choice** On what interval is the function $g(x) = e^{x^3-6x^2+8}$ decreasing?

(A) $(-\infty, 2]$ (B) $[0, 4]$ (C) $[2, 4]$
(D) $(4, \infty)$ (E) no interval

**55. Multiple Choice** Which of the following functions is an antiderivative of $\frac{1}{\sqrt{x}}$?

(A) $-\frac{1}{\sqrt{2x^3}}$ (B) $-\frac{2}{\sqrt{x}}$ (C) $\frac{\sqrt{x}}{2}$
(D) $\sqrt{x} + 5$ (E) $2\sqrt{x} - 10$

**56. Multiple Choice** All of the following functions satisfy the conditions of the Mean Value Theorem on the interval $[-1, 1]$ *except*

(A) $\sin x$ (B) $\sin^{-1} x$ (C) $x^{5/3}$ (D) $x^{3/5}$ (E) $\frac{x}{x-2}$

## Explorations

**57. *Analyzing Derivative Data*** Assume that $f$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. The table gives some values of $f'(x)$.

| $x$ | $f'(x)$ | $x$ | $f'(x)$ |
|---|---|---|---|
| $-2$ | 7 | 0.25 | $-4.81$ |
| $-1.75$ | 4.19 | 0.5 | $-4.25$ |
| $-1.5$ | 1.75 | 0.75 | $-3.31$ |
| $-1.25$ | $-0.31$ | 1 | $-2$ |
| $-1$ | $-2$ | 1.25 | $-0.31$ |
| $-0.75$ | $-3.31$ | 1.5 | 1.75 |
| $-0.5$ | $-4.25$ | 1.75 | 4.19 |
| $-0.25$ | $-4.81$ | 2 | 7 |
| 0 | $-5$ | | |

(a) Estimate where $f$ is increasing, decreasing, and has local extrema.

(b) Find a quadratic regression equation for the data in the table and superimpose its graph on a scatter plot of the data.

(c) Use the model in part (b) for $f'$ and find a formula for $f$ that satisfies $f(0) = 0$.

**58. *Analyzing Motion Data*** Priya's distance $D$ in meters from a motion detector is given by the data in Table 5.1.

**TABLE 5.1 Motion Detector Data**

| $t$ (sec) | $D$ (m) | $t$ (sec) | $D$ (m) |
|---|---|---|---|
| 0.0 | 3.36 | 4.5 | 3.59 |
| 0.5 | 2.61 | 5.0 | 4.15 |
| 1.0 | 1.86 | 5.5 | 3.99 |
| 1.5 | 1.27 | 6.0 | 3.37 |
| 2.0 | 0.91 | 6.5 | 2.58 |
| 2.5 | 1.14 | 7.0 | 1.93 |
| 3.0 | 1.69 | 7.5 | 1.25 |
| 3.5 | 2.37 | 8.0 | 0.67 |
| 4.0 | 3.01 | | |

(a) Estimate when Priya is moving toward the motion detector; away from the motion detector.

(b) **Writing to Learn** Give an interpretation of any local extreme values in terms of this problem situation.

(c) Find a cubic regression equation $D = f(t)$ for the data in Table 5.1 and superimpose its graph on a scatter plot of the data.

(d) Use the model in (c) for $f$ to find a formula for $f'$. Use this formula to estimate the answers to (a).

## Extending the Ideas

**59. *Geometric Mean*** The **geometric mean** of two positive numbers $a$ and $b$ is $\sqrt{ab}$. Show that for $f(x) = 1/x$ on any interval $[a, b]$ of positive numbers, the value of $c$ in the conclusion of the Mean Value Theorem is $c = \sqrt{ab}$.

**60. *Arithmetic Mean*** The **arithmetic mean** of two numbers $a$ and $b$ is $(a + b)/2$. Show that for $f(x) = x^2$ on any interval $[a, b]$, the value of $c$ in the conclusion of the Mean Value Theorem is $c = (a + b)/2$.

**61. *Upper Bounds*** Show that for any numbers $a$ and $b$ $|\sin b - \sin a| \le |b - a|$.

**62. *Sign of f'*** Assume that $f$ is differentiable on $a \le x \le b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

**63. *Monotonic Functions*** Show that monotonic increasing and decreasing functions are one-to-one.



# 5.3 Connecting $f'$ and $f''$ with the Graph of $f$

**What you will learn about . . .**

- First Derivative Test for Local Extrema
- Concavity
- Points of Inflection
- Second Derivative Test for Local Extrema
- Learning About Functions from Derivatives

**and why . . .**

Differential calculus is a powerful problem-solving tool precisely because of its usefulness for analyzing functions.

## First Derivative Test for Local Extrema

As we see once again in Figure 5.18, a function $f$ may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of $f'$ in a critical point's immediate vicinity. As $x$ moves from left to right, the values of $f$ increase where $f' > 0$ and decrease where $f' < 0$.

At the points where $f$ has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where $f$ has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

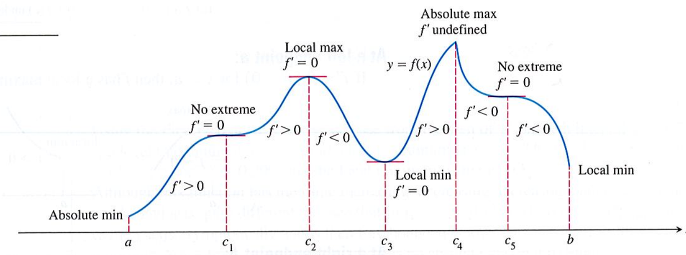


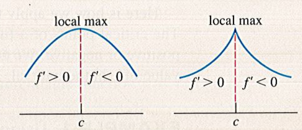

**Figure 5.18** A function's first derivative tells how the graph rises and falls.

**THEOREM 4 First Derivative Test for Local Extrema**

The following test applies to a continuous function $f(x)$.

**At a critical point $c$:**

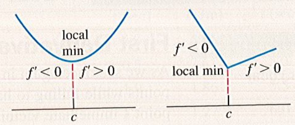
1. If $f'$ changes sign from positive to negative at $c$ ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then $f$ has a local maximum value at $c$.




(a) $f'(c) = 0$ (b) $f'(c)$ undefined

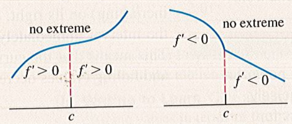
*continued*

2. If $f'$ changes sign from negative to positive at $c$ ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$) then $f$ has a local minimum value at $c$.




(a) $f'(c) = 0$ (b) $f'(c)$ undefined

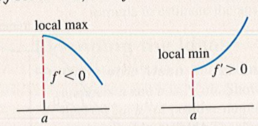
3. If $f'$ does not change sign at $c$ ($f'$ has the same sign on both sides of $c$), then $f$ has no local extreme value at $c$.




(a) $f'(c) = 0$ (b) $f'(c)$ undefined

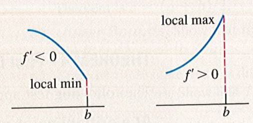
**At a left endpoint *a*:**

If $f' < 0$ ($f' > 0$) for $x > a$, then $f$ has a local maximum (minimum) value at $a$.




**At a right endpoint *b*:**

If $f' < 0$ ($f' > 0$) for $x < b$, then $f$ has a local minimum (maximum) value at $b$.




Here is how we apply the First Derivative Test to find the local extrema of a function. The critical points of a function $f$ partition the $x$-axis into intervals on which $f'$ is either positive or negative. We determine the sign of $f'$ in each interval by evaluating $f'$ for one value of $x$ in the interval. Then we apply Theorem 4 as shown in Examples 1 and 2.

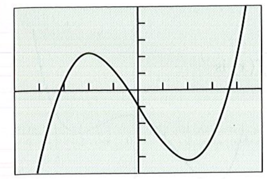
**EXAMPLE 1 Using the First Derivative Test**

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$ (b) $g(x) = (x^2 - 3)e^x$

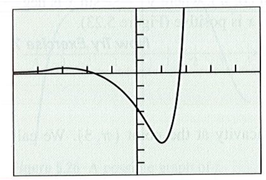
*continued*





[–5, 5] by [–25, 25]

**Figure 5.19** The graph of $f(x) = x^3 - 12x - 5$.



[–5, 5] by [–8, 5]

**Figure 5.20** The graph of $g(x) = (x^2 - 3)e^x$.

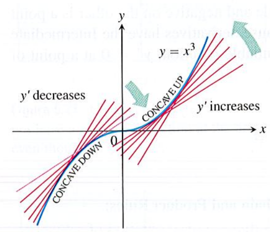
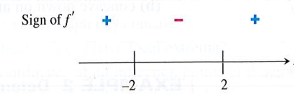



**Figure 5.21** The graph of $y = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

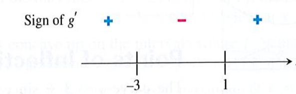
**SOLUTION**

(a) Since $f$ is differentiable for all real numbers, the only possible critical points are the zeros of $f'$. Solving $f'(x) = 3x^2 - 12 = 0$, we find the zeros to be $x = 2$ and $x = -2$. The zeros partition the $x$-axis into three intervals, as shown below:




Using the First Derivative Test, we can see from the sign of $f'$ on each interval that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The local maximum value is $f(-2) = 11$, and the local minimum value is $f(2) = -21$. There are no absolute extrema, as the function has range $(-\infty, \infty)$ (Figure 5.19).

(b) Since $g$ is differentiable for all real numbers, the only possible critical points are the zeros of $g'$. Since $g'(x) = (x^2 - 3) \cdot e^x + (2x) \cdot e^x = (x^2 + 2x - 3) \cdot e^x$, we find the zeros of $g'$ to be $x = 1$ and $x = -3$. The zeros partition the $x$-axis into three intervals, as shown below:




Using the First Derivative Test, we can see from the sign of $f'$ on each interval that there is a local maximum at $x = -3$ and a local minimum at $x = 1$. The local maximum value is $g(-3) = 6e^{-3} \approx 0.299$, and the local minimum value is $g(1) = -2e \approx -5.437$. Although this function has the same increasing–decreasing–increasing pattern as $f$, its left end behavior is quite different. We see that $\lim_{x \to -\infty} g(x) = 0$, so the graph approaches the $y$-axis asymptotically and is therefore bounded below. This makes $g(1)$ an *absolute* minimum. Since $\lim_{x \to \infty} g(x) = \infty$, there is no absolute maximum (Figure 5.20).

***Now Try Exercise 3.***

## Concavity

As you can see in Figure 5.21, the function $y = x^3$ rises as $x$ increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ *turn* in different ways. Looking at tangents as we scan from left to right, we see that the slope $y'$ of the curve decreases on the interval $(-\infty, 0)$ and then increases on the interval $(0, \infty)$. The curve $y = x^3$ is *concave down* on $(-\infty, 0)$ and *concave up* on $(0, \infty)$. The curve lies below the tangents where it is concave down, and above the tangents where it is concave up.

**DEFINITION Concavity**

The graph of a differentiable function $y = f(x)$ is

**(a)** **concave up** on an open interval $I$ if $y'$ is increasing on $I$.

**(b)** **concave down** on an open interval $I$ if $y'$ is decreasing on $I$.

If a function $y = f(x)$ has a second derivative, then we can conclude that $y'$ increases if $y'' > 0$ and $y'$ decreases if $y'' < 0$.

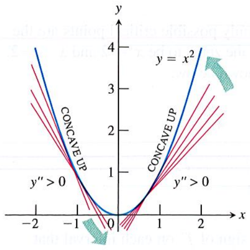
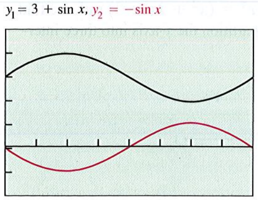



**Figure 5.22** The graph of $y = x^2$ is concave up on any interval. (Example 2)

$y_1 = 3 + \sin x$, $y_2 = -\sin x$



[0, 2π] by [−2, 5]

**Figure 5.23** Using the graph of $y''$ to determine the concavity of $y$. (Example 2)

### Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

**(a)** concave up on any interval where $y'' > 0$.

**(b)** concave down on any interval where $y'' < 0$.

### EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

**(a)** $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

**SOLUTION**

**(a)** Since $y'' = 2$ is always positive, the graph of $y = x^2$ is concave up on *any* interval. In particular, it is concave up on $(3, 10)$ (Figure 5.22).

**(b)** The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 5.23).

*Now Try Exercise 7.*

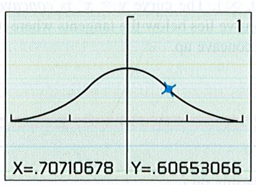
## Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

### DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

A point on a curve where $y''$ is positive on one side and negative on the other is a point of inflection. At such a point, $y''$ is either zero (because derivatives have the Intermediate Value Property) or undefined. If $y$ is a twice differentiable function, $y'' = 0$ at a point of inflection and $y'$ has a local maximum or minimum.




[−2, 2] by [−1, 2]

**Figure 5.24** Graphical confirmation that the graph of $y = e^{-x^2}$ has a point of inflection at $x = \sqrt{1/2}$ (and hence also at $x = -\sqrt{1/2}$). (Example 3)

### EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

**SOLUTION**

First we find the second derivative, recalling the Chain and Product Rules:

$$\begin{aligned} y &= e^{-x^2} \\ y' &= e^{-x^2} \cdot (-2x) \\ y'' &= e^{-x^2} \cdot (-2x) \cdot (-2x) + e^{-x^2} \cdot (-2) \\ &= e^{-x^2}(4x^2 - 2) \end{aligned}$$

The factor $e^{-x^2}$ is always positive, while the factor $(4x^2 - 2)$ changes sign at $-\sqrt{1/2}$ and at $\sqrt{1/2}$. Since $y''$ must also change sign at these two numbers, the points of inflection are $(-\sqrt{1/2}, 1/\sqrt{e})$ and $(\sqrt{1/2}, 1/\sqrt{e})$. We confirm our solution graphically by observing the changes of curvature in Figure 5.24.

*Now Try Exercise 13.*



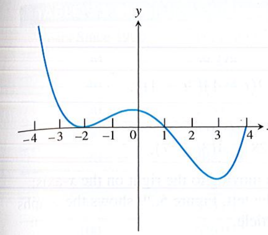


**Figure 5.25** The graph of $f'$ the derivative of $f$, on the interval $[-4, 4]$.

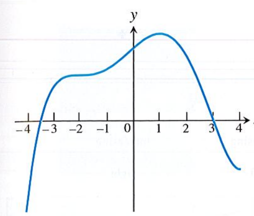
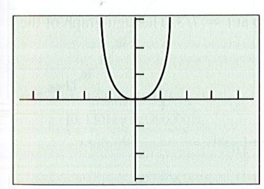
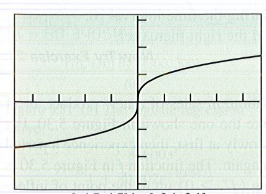


**Figure 5.26** A possible graph of $f$. (Example 4)



[−4.7, 4.7] by [−3.1, 3.1]

**Figure 5.27** The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[−4.7, 4.7] by [−3.1, 3.1]

**Figure 5.28** The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

### EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function $f$ on the interval $[-4, 4]$ is shown in Figure 5.25. Answer the following questions about $f$, justifying each answer with information obtained from the graph of $f'$.

**(a)** On what intervals is $f$ increasing?

**(b)** On what intervals is the graph of $f$ concave up?

**(c)** At which $x$-coordinates does $f$ have local extrema?

**(d)** What are the $x$-coordinates of all inflection points of the graph of $f$?

**(e)** Sketch a possible graph of $f$ on the interval $[-4, 4]$.

**SOLUTION**

Often, making a chart showing where $f'$ is positive and negative and where $f'$ is increasing and decreasing helps to understand the behavior of the function $f$ (whose derivative is $f'$). The following chart is based on Figure 5.25.

| Intervals | $-4 \le x < -2$ | $-2 < x \le 0$ | $0 < x \le 1$ | $1 < x \le 3$ | $3 < x \le 4$ |
|---|---|---|---|---|---|
| Sign of $f'$ | positive | positive | positive | negative | negative |
| Graph of $f'$ | decreasing | increasing | decreasing | decreasing | increasing |

**(a)** Since $f' > 0$ on the intervals $[-4, -2)$ and $(-2, 1)$, the function $f$ must be increasing on the entire interval $[-4, 1]$ with a horizontal tangent at $x = -2$ (a "shelf point").

**(b)** The graph of $f$ is concave up on the intervals where $f'$ is increasing. We see from the graph that $f'$ is increasing on the intervals $(-2, 0)$ and $(3, 4)$.

**(c)** By the First Derivative Test, there is a local maximum at $x = 1$ because the sign of $f'$ changes from positive to negative there. Note that there is no extremum at $x = -2$, since $f'$ does not change sign. Because the function increases from the left endpoint and decreases to the right endpoint, there are local minima at the endpoints $x = -4$ and $x = 4$.

**(d)** The inflection points of the graph of $f$ have the same $x$-coordinates as the turning points of the graph of $f'$, namely $-2$, 0, and 3.

**(e)** A possible graph satisfying all the conditions is shown in Figure 5.26.

*Now Try Exercise 23.*

*Caution:* It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

1. *The second derivative can be zero at a noninflection point.* For example, consider the function $f(x) = x^4$ (Figure 5.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that $f'$ does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point.* For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 5.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points of inflection.

### EXAMPLE 5 Studying Motion Along a Line

A particle is moving along the $x$-axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \ge 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

*continued*

## SOLUTION

### Solve Analytically

The velocity is

$$v(t) = x'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = x''(t) = 12t - 28 = 4(3t - 7).$$

When the function $x(t)$ is increasing, the particle is moving to the right on the $x$-axis; when $x(t)$ is decreasing, the particle is moving to the left. Figure 5.29 shows the graphs of the position, velocity, and acceleration of the particle.

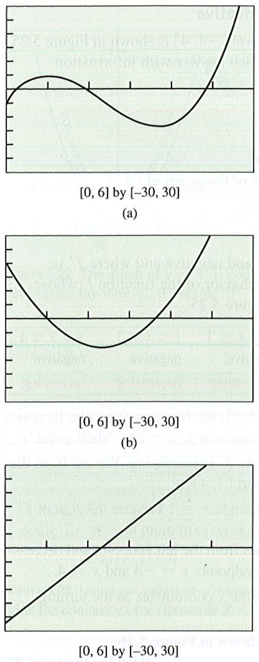
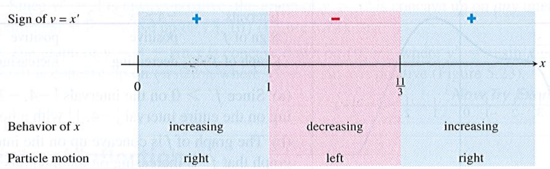


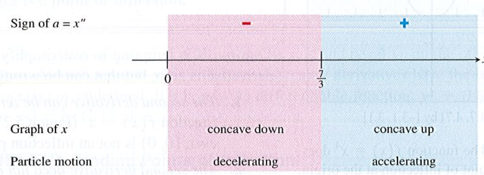

**Figure 5.29** The graph of
(a) $x(t) = 2t^3 - 14t^2 + 22t - 5, t \geq 0$,
(b) $x'(t) = 6t^2 - 28t + 22$, and
(c) $x''(t) = 12t - 28$. (Example 5)

Notice that the first derivative ($v = x'$) is zero when $t = 1$ and $t = 11/3$. These zeros partition the $t$-axis into three intervals, as shown in the sign graph of $v$ below:




The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$ and moving to the left in $(1, 11/3)$.

The acceleration $a(t) = 12t - 28$ has a single zero at $t = 7/3$. The sign graph of the acceleration is shown below:




The accelerating force is directed toward the left during the time interval $[0, 7/3]$, is momentarily zero at $t = 7/3$, and is directed toward the right thereafter.

***Now Try Exercise 25.***

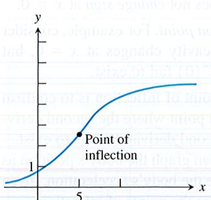



**Figure 5.30** A logistic curve

$$y = \frac{c}{1 + ae^{-bx}}.$$

The growth of an individual company, of a population, in sales of a new product, or of salaries often follows a ***logistic*** or ***life cycle curve*** like the one shown in Figure 5.30. For example, sales of a new product will generally grow slowly at first, then experience a period of rapid growth. Eventually, sales growth slows down again. The function $f$ in Figure 5.30 is increasing. Its rate of increase, $f'$, is at first increasing ($f'' > 0$) up to the point of inflection, and then its rate of increase, $f'$, is decreasing ($f'' < 0$). This is, in a sense, the opposite of what happens in Figure 5.21.

Some graphers have the logistic curve as a built-in regression model. We use this feature in Example 6.



**TABLE 5.2 Population of Alaska**

| Years Since 1900 | Population |
|---|---|
| 30 | 59,278 |
| 40 | 75,524 |
| 50 | 128,643 |
| 60 | 226,167 |
| 70 | 302,583 |
| 80 | 401,851 |
| 90 | 550,043 |
| 100 | 626,932 |
| 109 | 698,473 |

*Source:* Bureau of the Census, U.S. Chamber of Commerce; www.quickfacts.census.gov.

### EXAMPLE 6 Population Growth in Alaska

Table 5.2 shows the population of Alaska from 1930 to 2009 in years since 1900.

**(a)** Find the logistic regression for the data.

**(b)** Use the regression equation to predict the Alaskan population in the 2020 census.

**(c)** Find the inflection point of the regression equation. What significance does the inflection point have in terms of population growth in Alaska?

**(d)** What does the regression equation indicate about the population of Alaska in the long run?

## SOLUTION

**(a)** Using years since 1900 as the independent variable and population as the dependent variable, the logistic regression equation is approximately

$$y = \frac{846003}{1 + 80.902e^{-0.0547x}}$$

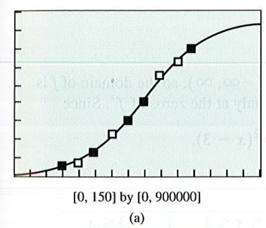
Its graph is superimposed on a scatter plot of the data in Figure 5.31(a). Store the regression equation as Y1 in your calculator.

**(b)** The calculator reports Y1(120) to be approximately 759,204.094. (Given the uncertainty of this kind of extrapolation, it is probably more reasonable to say "approximately 759,200.")

**(c)** The inflection point will occur where $y''$ changes sign. Finding $y''$ algebraically would be tedious, but we can graph the numerical derivative of the numerical derivative and find the zero graphically. Figure 5.31(b) shows the graph of $y''$, which is nDeriv(nDeriv(Y1,X,X),X,X) in calculator syntax. The zero is approximately 80, so the inflection point occurred in 1980, when the population was about 402,000 and growing the fastest.

**(d)** Notice that $\lim_{x \to \infty} \frac{846003}{1 + 80.902e^{-0.0547x}} = 846003$, so the regression equation indicates that the population of Alaska will stabilize at about 846,003 in the long run. Do not put too much faith in this number, however, as human population is dependent on too many variables that can, and will, change over time.

***Now Try Exercise 31.***



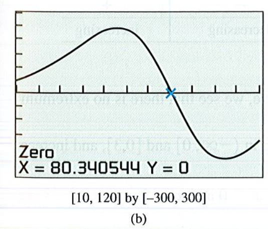




**Figure 5.31** (a) The logistic regression curve

$$y = \frac{846003}{1 + 80.902e^{-0.0547x}}$$

superimposed on the population data from Table 5.2, and (b) the graph of $y''$ showing a zero at about $x = 80.341$.

## Second Derivative Test for Local Extrema

Instead of looking for sign changes in $y'$ at critical points, we can sometimes use the following test to determine the presence of local extrema.

**THEOREM 5 Second Derivative Test for Local Extrema**

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.

This test requires us to know $f''$ *only at c itself* and not in an interval about $c$. This makes the test easy to apply. That's the good news. The bad news is that the test fails if $f''(c) = 0$ or if $f''(c)$ fails to exist. When this happens, go back to the First Derivative Test for local extreme values.

In Example 7, we apply the Second Derivative Test to the function in Example 1.

### EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

**SOLUTION**

We have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$f''(x) = 6x.$$

Testing the critical points $x = \pm 2$ (there are no endpoints), we find

$$f''(-2) = -12 < 0 \Rightarrow f \text{ has a local maximum at } x = -2 \text{ and}$$
$$f''(2) = 12 > 0 \Rightarrow f \text{ has a local minimum at } x = 2.$$

*Now Try Exercise 35.*

### EXAMPLE 8 Using $f'$ and $f''$ to Graph $f$

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of $f$ occur.

(b) Find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing.

(c) Find where the graph of $f$ is concave up and where it is concave down.

(d) Sketch a possible graph for $f$.

**SOLUTION**

$f$ is continuous since $f'$ exists. The domain of $f'$ is $(-\infty, \infty)$, so the domain of $f$ is also $(-\infty, \infty)$. Thus, the critical points of $f$ occur only at the zeros of $f'$. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$.

| Intervals | $x < 0$ | $0 < x < 3$ | $3 < x$ |
|---|---|---|---|
| Sign of $f'$ | $-$ | $-$ | $+$ |
| Behavior of $f$ | decreasing | decreasing | increasing |

(a) Using the First Derivative Test and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that $f$ is decreasing in $(-\infty, 0]$ and $[0,3]$, and increasing in $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

| Intervals | $x < 0$ | $0 < x < 2$ | $2 < x$ |
|---|---|---|---|
| Sign of $f''$ | $+$ | $-$ | $+$ |
| Behavior of $f$ | concave up | concave down | concave up |

We see that $f$ is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

*continued*

**Note**

The *Second Derivative Test* does not apply at $x = 0$ because $f''(0) = 0$. We need the First Derivative Test to see that there is no local extremum at $x = 0$.



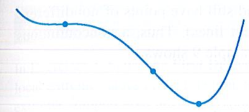


**Figure 5.32** The graph for $f$ has no extremum but has points of inflection where $x = 0$ and $x = 2$, and a local minimum where $x = 3$. (Example 8)

(d) Summarizing the information in the two tables above, we obtain

| $x < 0$ | $0 < x < 2$ | $2 < x < 3$ | $x < 3$ |
|---|---|---|---|
| decreasing | decreasing | decreasing | increasing |
| concave up | concave down | concave up | concave up |

Figure 5.32 shows one possibility for the graph of $f$.

*Now Try Exercise 39*

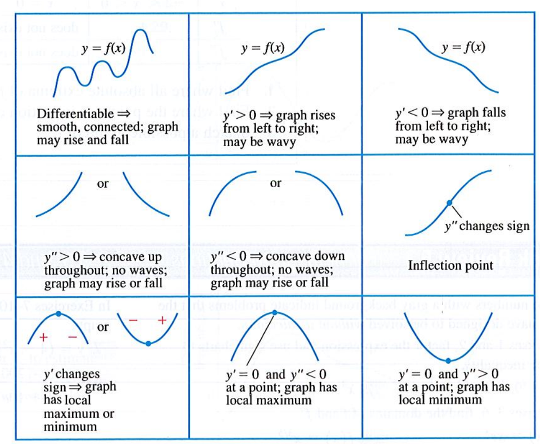
### EXPLORATION 1 Finding $f$ from $f'$

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

## Learning About Functions from Derivatives

We have seen in Example 8 and Exploration 1 that we are able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining $y'$. We can find where the graph rises and falls and where any local extrema are assumed. We can differentiate $y'$ to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the $xy$-plane. As we discovered in Section 5.2, the only additional information we need to position the graph is the value of $f$ at one point.



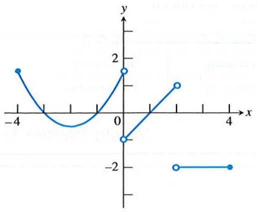



**Figure 5.33** The graph of $f'$, a discontinuous derivative.

Remember also that a function can be continuous and still have points of nondifferentiability (cusps, corners, and points with vertical tangent lines). Thus, a noncontinuous graph of $f'$ could lead to a continuous graph of $f$, as Example 9 shows.

### EXAMPLE 9 Analyzing a Discontinuous Derivative

A function $f$ is continuous on the interval $[-4, 4]$. The discontinuous function $f'$, with domain $[-4, 0) \cup (0, 2) \cup (2, 4)]$, is shown in the graph to the left (Figure 5.33).

(a) Find the $x$-coordinates of all local extrema and points of inflection of $f$.

(b) Sketch a possible graph of $f$.

### SOLUTION

**(a)** For extrema, we look for places where $f'$ changes sign. There are local maxima at $x = -3$, 0, and 2 (where $f'$ goes from positive to negative) and local minima at $x = -1$ and 1 (where $f'$ goes from negative to positive). There are also local minima at the two endpoints $x = -4$ and 4, because $f'$ starts positive at the left endpoint and ends negative at the right endpoint.

For points of inflection, we look for places where $f''$ changes sign, that is, where the graph of $f'$ changes direction. This occurs only at $x = -2$.

**(b)** A possible graph of $f$ is shown in Figure 5.34. The derivative information determines the shape of the three components, and the continuity condition determines that the three components must be linked together.

***Now Try Exercises 49 and 53.***

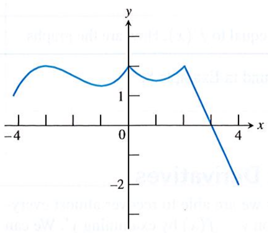



**Figure 5.34** A possible graph of $f$. (Example 9)

### EXPLORATION 2 Finding $f$ from $f'$ and $f''$

A function $f$ is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and $f'$ and $f''$ have the following properties.

| $x$ | $-2 < x < 0$ | $x = 0$ | $0 < x < 2$ | $x = 2$ | $2 < x < 4$ |
|---|---|---|---|---|---|
| $f'$ | + | does not exist | – | 0 | – |
| $f''$ | + | does not exist | + | 0 | – |

1. Find where all absolute extrema of $f$ occur.
2. Find where the points of inflection of $f$ occur.
3. Sketch a possible graph of $f$.

**Note**

Because sign charts are very helpful, you will want to use and analyze them on a regular basis.

## Quick Review 5.3 *(For help, go to Sections 1.3, 2.2, 3.3, and 4.4.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1 and 2, factor the expression and use sign charts to solve the inequality.

**1.** $x^2 - 9 < 0$ **2.** $x^3 - 4x > 0$

In Exercises 3–6, find the domains of $f$ and $f'$.

**3.** $f(x) = xe^x$ **4.** $f(x) = x^{3/5}$

**5.** $f(x) = \dfrac{x}{x-2}$ **6.** $f(x) = x^{2/5}$

In Exercises 7–10, find the horizontal asymptotes of the function's graph.

**7.** $y = (4 - x^2)e^x$ **8.** $y = (x^2 - x)e^{-x}$

**9.** $y = \dfrac{200}{1 + 10e^{-0.5x}}$ **10.** $y = \dfrac{750}{2 + 5e^{-0.1x}}$



## Section 5.3 Exercises

In Exercises 1–6, use the **First Derivative Test** to determine the local extreme values of the function, and identify any absolute extrema. Support your answers graphically.

**1.** $y = x^2 - x - 1$ **2.** $y = -2x^3 + 6x^2 - 3$

**3.** $y = 2x^4 - 4x^2 + 1$ **4.** $y = xe^{1/x}$

**5.** $y = x\sqrt{8 - x^2}$ **6.** $y = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 7–12, use the Concavity Test to determine the intervals on which the graph of the function is **(a)** concave up and **(b)** concave down.

**7.** $y = 4x^3 + 21x^2 + 36x - 20$

**8.** $y = -x^4 + 4x^3 - 4x + 1$

**9.** $y = 2x^{1/5} + 3$ **10.** $y = 5 - x^{1/3}$

**11.** $y = \begin{cases} 2x, & x < 1 \\ 2 - x^2, & x \geq 1 \end{cases}$ **12.** $y = e^x, \quad 0 \leq x \leq 2\pi$

In Exercises 13 –20, find all points of inflection of the function.

**13.** $y = xe^x$ **14.** $y = x\sqrt{9 - x^2}$

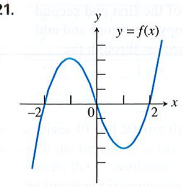
**15.** $y = \tan^{-1} x$ **16.** $y = x^3(4 - x)$

**17.** $y = x^{1/3}(x - 4)$ **18.** $y = x^{1/2}(x + 3)$

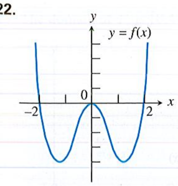
**19.** $y = \dfrac{x^3 - 2x^2 + x - 1}{x - 2}$ **20.** $y = \dfrac{x}{x^2 + 1}$

In Exercises 21 and 22, use the graph of the function $f$ to estimate where **(a)** $f'$ and **(b)** $f''$ are 0, positive, and negative.

**21.**

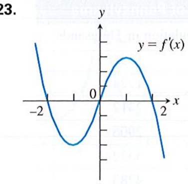



**22.**

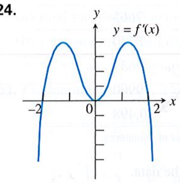



In Exercises 23 and 24, use the graph of the function $f'$ to estimate the intervals on which the function $f$ is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the $x$-coordinates of all local extreme values.

**23.**




**24.**



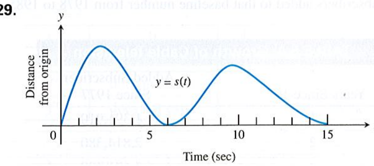

In Exercises 25–28, a particle is moving along the $x$-axis with position function $x(t)$. Find the **(a)** velocity and **(b)** acceleration, and **(c)** describe the motion of the particle for $t \geq 0$.

**25.** $x(t) = t^2 - 4t + 3$ **26.** $x(t) = 6 - 2t - t^2$

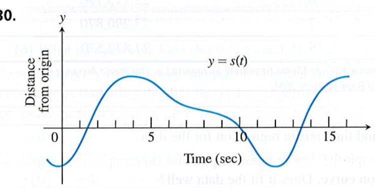
**27.** $x(t) = t^3 - 3t + 3$ **28.** $x(t) = 3t^2 - 2t^3$

In Exercises 29 and 30, the graph of the position function $y = s(t)$ of a particle moving along a line is given. At approximately what times is the particle's **(a)** velocity equal to zero? **(b)** acceleration equal to zero?

**29.**




**30.**



**31.** Table 5.3 shows the population of Pennsylvania in each 10-year census between 1830 and 1950.

**TABLE 5.3 Population of Pennsylvania**

| Years Since 1820 | Population in Thousands |
|---|---|
| 10 | 1348 |
| 20 | 1724 |
| 30 | 2312 |
| 40 | 2906 |
| 50 | 3522 |
| 60 | 4283 |
| 70 | 5258 |
| 80 | 6302 |
| 90 | 7665 |
| 100 | 8720 |
| 110 | 9631 |
| 120 | 9900 |
| 130 | 10,498 |

*Source:* Bureau of the Census, U.S. Chamber of Commerce.

**(a)** Find the logistic regression for the data.

**(b)** Graph the data in a scatter plot and superimpose the regression curve.

**(c)** Use the regression equation to predict the Pennsylvania population in the 2000 census.

**(d)** In what year was the Pennsylvania population growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

**(e)** What does the regression equation indicate about the population of Pennsylvania in the long run?

**32.** In 1977, there were 12,168,450 basic cable television subscribers in the United States. Table 5.4 shows the cumulative number of subscribers added to that baseline number from 1978 to 1985.

**TABLE 5.4 Growth of Cable Television**

| Years Since 1977 | Added Subscribers Since 1977 |
|---|---|
| 1 | 1,391,910 |
| 2 | 2,814,380 |
| 3 | 5,671,490 |
| 4 | 11,219,200 |
| 5 | 17,340,570 |
| 6 | 22,113,790 |
| 7 | 25,290,870 |
| 8 | 27,872,520 |

*Source:* Nielsen Media Research, as reported in *The World Almanac and Book of Facts 2004.*

**(a)** Find the logistic regression for the data.

**(b)** Graph the data in a scatter plot and superimpose the regression curve. Does it fit the data well?

**(c)** In what year between 1977 and 1985 were basic cable TV subscriptions growing the fastest? What significant behavior does the graph of the regression equation exhibit at that point?

**(d)** What does the regression equation indicate about the number of basic cable television subscribers in the long run? (Be sure to add the baseline 1977 number.)

**(e) Writing to Learn** In fact, the long-run number of basic cable subscribers predicted by the regression equation falls short of the actual 2010 number by more than 20 million. What circumstances changed to render the earlier model so ineffective?

In Exercises 33–38, use the Second Derivative Test to find the local extrema for the function.

**33.** $y = 3x - x^3 + 5$

**34.** $y = x^5 - 80x + 100$

**35.** $y = x^3 + 3x^2 - 2$

**36.** $y = 3x^5 - 25x^3 + 60x + 20$

**37.** $y = xe^x$

**38.** $y = xe^{-x}$

In Exercises 39 and 40, use the derivative of the function $y = f(x)$ to find the points at which $f$ has a

**(a)** local maximum, **(b)** local minimum, or

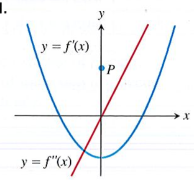
**(c)** point of inflection.

**39.** $y' = (x - 1)^2(x - 2)$

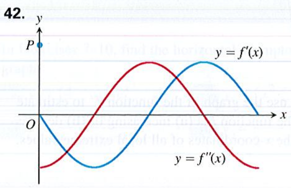
**40.** $y' = (x - 1)^2(x - 2)(x - 4)$

Exercises 41 and 42 show the graphs of the first and second derivatives of a function $y = f(x)$. Copy the figure and add a sketch of a possible graph of $f$ that passes through the point $P$.

**41.**




**42.**






**43. Writing to Learn** If $f(x)$ is a differentiable function and $f'(c) = 0$ at an interior point $c$ of $f$'s domain, must $f$ have a local maximum or minimum at $x = c$? Explain.

**44. Writing to Learn** If $f(x)$ is a twice-differentiable function and $f''(c) = 0$ at an interior point $c$ of $f$'s domain, must $f$ have an inflection point at $x = c$? Explain.

**45. *Connecting f and f′*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

**46. *Connecting f and f″*** Sketch a smooth curve $y = f(x)$ through the origin with the properties that $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

**47. *Connecting f, f′, and f″*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

$f(-2) = 8$    $f'(x) > 0$ for $|x| > 2$

$f(0) = 4$    $f'(x) < 0$ for $|x| < 2$

$f(2) = 0$    $f''(x) < 0$ for $x < 0$

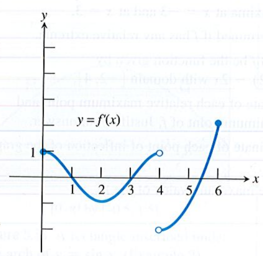
$f'(2) = f'(-2) = 0$    $f''(x) > 0$ for $x > 0$

**48. *Using Behavior to Sketch*** Sketch a continuous curve $y = f(x)$ with the following properties. Label coordinates where possible.

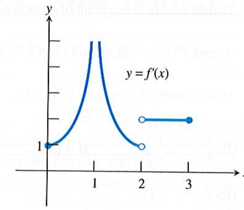
| $x$ | $y$ | Curve |
|---|---|---|
| $x < 2$ | | falling, concave up |
| 2 | 1 | horizontal tangent |
| $2 < x < 4$ | | rising, concave up |
| 4 | 4 | inflection point |
| $4 < x < 6$ | | rising, concave down |
| 6 | 7 | horizontal tangent |
| $x > 6$ | | falling, concave down |

In Exercises 49 and 50, use the graph of $f'$ to estimate the intervals on which the function $f$ is **(a)** increasing or **(b)** decreasing. Also, **(c)** estimate the $x$-coordinates of all local extreme values. (Assume that the function $f$ is continuous, even at the points where $f'$ is undefined.)

**49.** The domain of $f'$ is $[0,4) \cup (4,6]$.




**50.** The domain of $f'$ is $[0,1) \cup (1,2) \cup (2,3]$.




**Group Activity** In Exercises 51 and 52, do the following.

**(a)** Find the absolute extrema of $f$ and where they occur.

**(b)** Find any points of inflection.

**(c)** Sketch a possible graph of $f$.

**51.** $f$ is continuous on $[0, 3]$ and satisfies the following.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f$ | 0 | 2 | 0 | $-2$ |
| $f'$ | 3 | 0 | does not exist | $-3$ |
| $f''$ | 0 | $-1$ | does not exist | 0 |

| $x$ | $0 < x < 1$ | $1 < x < 2$ | $2 < x < 3$ |
|---|---|---|---|
| $f$ | + | + | − |
| $f'$ | + | − | − |
| $f''$ | − | − | − |

**52.** $f$ is an even function, continuous on $[-3, 3]$, and satisfies the following.

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f$ | 2 | 0 | $-1$ |
| $f'$ | does not exist | 0 | does not exist |
| $f''$ | does not exist | 0 | does not exist |

| $x$ | $0 < x < 1$ | $1 < x < 2$ | $2 < x < 3$ |
|---|---|---|---|
| $f$ | + | − | − |
| $f'$ | − | − | + |
| $f''$ | + | − | − |

**(d)** What can you conclude about $f(3)$ and $f(-3)$?

**Group Activity** In Exercises 53 and 54, sketch a possible graph of a continuous function $f$ that has the given properties.

**53.** Domain $[0, 6]$, graph of $f'$ given in Exercise 49, and $f(0) = 2$.

**54.** Domain $[0, 3]$, graph of $f'$ given in Exercise 50, and $f(0) = -3$.

## Standardized Test Questions

**55. True or False** If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection. Justify your answer.

**56. True or False** If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Justify your answer.

**57. Multiple Choice** If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on

(A) $\left(-\infty, -\frac{1}{a}\right)$ (B) $\left(-\infty, \frac{1}{a}\right)$ (C) $\left(-\frac{1}{a}, \infty\right)$

(D) $\left(\frac{1}{a}, \infty\right)$ (E) $(-\infty, -1)$

**58. Multiple Choice** If $f(0) = f'(0) = f''(0) = 0$, which of the following *must be true*?

(A) There is a local maximum of $f$ at the origin.

(B) There is a local minimum of $f$ at the origin.

(C) There is no local extremum of $f$ at the origin.

(D) There is a point of inflection of the graph of $f$ at the origin.

(E) There is a horizontal tangent to the graph of $f$ at the origin.

**59. Multiple Choice** The $x$-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are

(A) 0 only (B) 1 only (C) 3 only (D) 0 and 3 (E) 0 and 1

**60. Multiple Choice** Which of the following conditions would enable you to conclude that the graph of $f$ has a point of inflection at $x = c$?

(A) There is a local maximum of $f'$ at $x = c$.

(B) $f''(c) = 0$.

(C) $f''(c)$ does not exist.

(D) The sign of $f'$ changes at $x = c$.

(E) $f$ is a cubic polynomial and $c = 0$.

## Exploration

**61. *Graphs of Cubics*** There is almost no leeway in the locations of the inflection point and the extrema of $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, because the one inflection point occurs at $x = -b/(3a)$ and the extrema, if any, must be located symmetrically about this value of $x$. Check this out by examining **(a)** the cubic in Exercise 7 and **(b)** the cubic in Exercise 2. Then **(c)** prove the general case.

## Extending the Ideas

In Exercises 62 and 63, feel free to use a CAS (computer algebra system), if you have one, to solve the problem.

**62. *Logistic Functions*** Let $f(x) = c/(1 + ae^{-bx})$ with $a > 0$, $abc \neq 0$.

**(a)** Show that $f$ is increasing on the interval $(-\infty, \infty)$ if $abc > 0$, and decreasing if $abc < 0$.

**(b)** Show that the point of inflection of $f$ occurs at $x = (\ln|a|)/b$.

**63. *Quartic Polynomial Functions*** Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$.

**(a)** Show that the graph of $f$ has 0 or 2 points of inflection.

**(b)** Write a condition that must be satisfied by the coefficients if the graph of $f$ has 0 or 2 points of inflection.

## Quick Quiz for AP* Preparation: Sections 5.1–5.3

**1. Multiple Choice** How many critical points does the function $f(x) = (x - 2)^5(x + 3)^4$ have?

(A) One (B) Two (C) Three (D) Five (E) Nine

**2. Multiple Choice** For what value of $x$ does the function $f(x) = (x - 2)(x - 3)^2$ have a relative maximum?

(A) $-3$ (B) $-\frac{7}{3}$ (C) $-\frac{5}{2}$ (D) $\frac{7}{3}$ (E) $\frac{5}{2}$

**3. Multiple Choice** If $g$ is a differentiable function such that $g(x) < 0$ for all real numbers $x$, and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

(A) $f$ has a relative maximum at $x = -3$ and a relative minimum at $x = 3$.

(B) $f$ has a relative minimum at $x = -3$ and a relative maximum at $x = 3$.

(C) $f$ has relative minima at $x = -3$ and at $x = 3$.

(D) $f$ has relative maxima at $x = -3$ and at $x = 3$.

(E) It cannot be determined if $f$ has any relative extrema.

**4. Free Response** Let $f$ be the function given by $f(x) = 3\ln(x^2 + 2) - 2x$ with domain $[-2, 4]$.

**(a)** Find the coordinate of each relative maximum point and each relative minimum point of $f$. Justify your answer.

**(b)** Find the $x$-coordinate of each point of inflection of the graph of $f$.

**(c)** Find the absolute maximum value of $f(x)$.



# 5.4 Modeling and Optimization

**What you will learn about . . .**

- Examples from Mathematics
- Examples from Business and Industry
- Examples from Economics
- Modeling Discrete Phenomena with Differentiable Functions

**and why . . .**

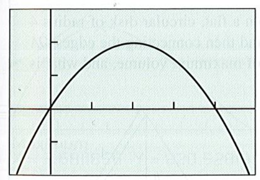
Historically, optimization problems were among the earliest applications of what we now call differential calculus.

## Examples from Mathematics

While today's graphing technology makes it easy to find extrema without calculus, the algebraic methods of differentiation were understandably more practical, and certainly more accurate, when graphs had to be rendered by hand. Indeed, one of the oldest applications of what we now call "differential calculus" (pre-dating Newton and Leibniz) was to find maximum and minimum values of functions by finding where horizontal tangent lines might occur. We will use both algebraic and graphical methods in this section to solve "max-min" problems in a variety of contexts, but the emphasis will be on the *modeling* process that both methods have in common. Here is a strategy you can use:

**Strategy for Solving Max-Min Problems**

1. **Understand the Problem** Read the problem carefully. Identify the information you need to solve the problem.
2. **Develop a Mathematical Model of the Problem** Draw pictures and label the parts that are important to the problem. Introduce a variable to represent the quantity to be maximized or minimized. Using that variable, write a function whose extreme value gives the information sought.
3. **Graph the Function** Find the domain of the function. Determine what values of the variable make sense in the problem.
4. **Identify the Critical Points and Endpoints** Find where the derivative is zero or fails to exist.
5. **Solve the Mathematical Model** If unsure of the result, support or confirm your solution with another method.
6. **Interpret the Solution** Translate your mathematical result into the problem setting and decide whether the result makes sense.



[–5, 25] by [–100, 150]

**Figure 5.35** The graph of $f(x) = x(20 - x)$ with domain $(-\infty, \infty)$ has an absolute maximum of 100 at $x = 10$. (Example 1)

**EXAMPLE 1 Using the Strategy**

Find two numbers whose sum is 20 and whose product is as large as possible.

**SOLUTION**

**Model** If one number is $x$, the other is $(20 - x)$, and their product is $f(x) = x(20 - x)$.

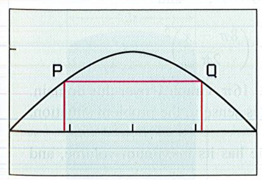
**Solve Graphically** We can see from the graph of $f$ in Figure 5.35 that there is a maximum. From what we know about parabolas, the maximum occurs at $x = 10$.

**Confirm Analytically** When $x = 10$, $f'(x) = 20 - 2x = 0$. Since $f''(x) = -2$ (always negative), the maximum occurs at $x = 10$. The other number is $20 - x = 10$.

**Interpret** The two numbers we seek are $x = 10$ and $20 - x = 10$.

*Now Try Exercise 1.*

Sometimes we find it helpful to use both analytic and graphical methods together, as in Example 2.




[0, π] by [–0.5, 1.5]

**Figure 5.36** A rectangle inscribed under one arch of $y = \sin x$. (Example 2)

**EXAMPLE 2 Inscribing Rectangles**

A rectangle is to be inscribed under one arch of the sine curve (Figure 5.36). What is the largest area the rectangle can have, and what dimensions give that area?

*continued*